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Blending two discrete integrability criteria: singularity confinement and algebraic entropy

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Abstract. We confront two integrability criteria for rational mappings. The first is the singularity confinement based on the requirement that every singularity, spontaneously appearing during the iteration of a mapping, disappear after some steps. The second recently proposed is the algebraic entropy criterion associated to the growth of the degree of the iterates. The algebraic entropy results confirm the previous findings of singularity confinement on discrete Painlevé equations. The degree-growth methods are also applied to linearisable systems. The result is that systems integrable through linearisation have a slower growth than systems integrable through isospectral methods. This may provide a valuable detector of not just integrability but also of the precise integration method. We propose an extension of the Gambier mapping in N dimensions. Finally a dual strategy for the investigation of the integrability of discrete systems is proposed based on both singularity confinement and the low growth requirement.

1. Introduction

The detection of discrete integrability has for the past few years relied on the criterion of singularity confinement [1]. This criterion is based on the observation that for integrable discrete systems any singularity that appears spontaneously disappears after some iterations. Let us give one example. In the mapping

$$x_{n+1} + x_{n-1} = \frac{a}{x_n} + \frac{1}{x_n^2}, \tag{1.1}$$

where a is constant, we assume that, at some iteration, x_{n-1} vanishes while x_{n-2} is finite. This has as consequence a diverging x_n, and iterating once more a vanishing x_{n+1}. If we try to compute x_{n+2} we obtain an indeterminate form, $\infty - \infty$. In order to lift the indeterminacy we introduce a small parameter ϵ and assume that $x_{n-1} = \epsilon$. Iterating we obtain successively, expanding in ϵ, $x_n = 1/\epsilon^2 + a/\epsilon - x_{n-2} + \mathcal{O}(\epsilon)$, $x_{n+1} = -\epsilon + a\epsilon^2 + \mathcal{O}(\epsilon^3)$, $x_{n+2} = x_{n-2} + 2(ax_{n-2} + 1)\epsilon + \mathcal{O}(\epsilon^2)$. By taking the appropriate limit $\epsilon \to 0$ we find that x_{n+2} is finite, and moreover contains the information on the initial data i.e. x_{n-2}. The subsequent x_n's are indeed

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finite and the singularity is confined to the sequence $\{0, \infty^2, 0\}$, the square in the infinite sign being a reminder of the $1/\epsilon^2$. Singularity confinement is tailored for rational mappings although this should not be considered as an absolute restriction. What singularity confinement *cannot* treat are mappings which do *not* possess singularities. For example a mapping of the form $x_{n+1} = \lambda x_n(1 - x_n)$ lies beyond the reach of the confinement method [2]. Our tacit conjecture has always been that polynomial mappings, like the logistic one, are *not* integrable, with only exception the linear one [3].

While confinement has turned out to be a most useful integrability detector, its application was based on the unwarranted assumption that its necessary character will be constraining enough to make it sufficient. Still, from the outset it was clear that confinement was not a sufficient criterion. Let us recall the example presented already in [4]. If we examine the singularity structure of the mapping $x_{n+1} = f(x_n, n)$ where the f is rational in x_n and analytic in n we find that all f's of the form $f = \sum_k \frac{\alpha_k}{(x_n+\beta_k)^{\nu_k}}$, with $\nu_k \in \mathbb{Z}$, lead to confinement. Clearly not all these mappings can be integrable and, indeed, only the homographic one, $x_{n+1} = \alpha + \frac{\lambda}{x_n+\beta}$, is. The non sufficient character of singularity confinement was put to a wider perspective by Hietarinta and Viallet [5] who verified that the mapping:

$$x_{n+1} + x_{n-1} = x_n + \frac{1}{x_n^2} \tag{1.2}$$

is confining, the singularity sequence being $\{0, \infty^2, \infty^2, 0\}$, and at the same time exhibits chaotic behaviour. Moreover this is not an isolated example and one can easily construct mappings which are confining without being integrable.

Clearly a more stringent integrability criterion was needed and the authors of [5] (see also [6]), have proposed one based on the ideas of Arnold and Veselov. According to Arnold [7] the complexity (in the case of mappings of the plane) is the number of intersection points of a fixed curve with the image of a second curve obtained under the mapping at hand. While the complexity grows exponentially with the iteration for generic mappings, it can be shown [8] to grow only polynomially for a large class of integrable mappings. As Veselov points out, "integrability has an essential correlation with the weak growth of certain characteristics". Thus the authors of [5] proposed to directly test the degree of the successive iterates and introduced the notion of algebraic entropy. It is defined as $E = \lim_{n\to\infty} \log(d_n)/n$ where d_n is the degree of the nth iterate of some initial data under the action of the mapping. Since a generic nonintegrable mapping exhibits exponential degree-growth, a nonzero algebraic entropy indicates nonintegrability. Integrable mappings must have zero algebraic entropy, associated to slower-than-exponential (typically polynomial) degree-growth. It is expected that the requirement of zero algebraic entropy is strong enough to be a sufficient integrability condition.

In order to make the ideas clearer we present here our method for the study of the degree-growth. It presents a few differences with respect to the one of [5] which, we feel, make its practical implementation simpler. We start by introducing homogeneous variables through the appropriate choice of initial data. Typically, in a three-point mapping we choose to introduce $x_0 = r$, $x_1 = p/q$ and thus the weight of r is 0 while that of p, q are taken equal to 1. We then compute the homogeneous degree in p, q in the denominator and numerator of x_n at every iteration. Let us follow the first few iterations of mapping (1.1) in order to understand the mechanism

of the degree-growth. We find

$$x_2 = \frac{q^2 + apq - rp^2}{p^2}, \quad x_3 = \frac{pP_4}{q(q^2 + apq - rp^2)^2},$$
$$x_4 = \frac{(q^2 + apq - rp^2)P_6}{P_4^2}, \quad x_5 = \frac{P_4 P_9}{qP_6^2}$$

where the P_k's are homogeneous polynomials in p, q of degree k. (Remember that r is of zero homogeneous degree in our convention). A pattern becomes apparent. Whenever a new polynomial appears in the numerator of x_n, its square will appear in the denominator of x_{n+1} and it will appear one last time as a factor of the numerator of x_{n+2}, after which it disappears due to factorisations. The singularities we are working with in the singularity confinement approach correspond to the zeros of any of these polynomials, which explains the pattern $\{0, \infty^2, 0\}$. The singularity confinement is intimately related to this factorisation which plays a crucial role in the algebraic entropy approach. If we calculate the degree of the iterates, we obtain: 0, 1, 2, 5, 8, 13, 18, 25, 32, 41, ... Clearly the degree-growth is polynomial: we have $d_{2m} = 2m^2$ and $d_{2m+1} = 2m^2 + 2m + 1$. (A remark is necessary at this point. In order to obtain a closed-form expression for the degrees of the iterates, we start by computing a sufficient number of them. Once the expression of the degree has been heuristically established we compute the next few ones and check that they agree with the analytical expression predicted). Thus the algebraic entropy of this mapping is zero in agreement with its integrable character [9].

On the other hand, if we iterate mapping (1.2) we obtain the sequence

$$x_2 = \frac{p^3 - p^2qr + q^3}{p^2q}, \quad x_3 = \frac{P_8}{p^2(p^3 - p^2qr + q^3)^2},$$
$$x_4 = \frac{pP_{22}}{q(p^3 - p^2qr + q^3)^2 {P_8}^2}, \quad x_5 = \frac{(p^3 - p^2qr + q^3)P_{58}}{q{P_8}^2{P_{22}}^2}.$$

Again some factorisations and simplifications do occur which explain why this mapping has the confinement property. However the degree of the new terms appearing at every iteration grows too rapidly and thus the simplifications cannot curb the exponential growth. The degree sequence now is $0, 1, 3, 8, 23, 61, 160, 421, \ldots$, i.e. the degrees obey the relation $d_{n+1} - 3d_n + d_{n-1} = \frac{2}{3}(1 + j^{n+1} + j^{2(n+1)})$ where j is a complex cubic root of 1 leading to an exponential growth with asymptotic ratio $(3 + \sqrt{5})/2$ and algebraic entropy $\log((3 + \sqrt{5})/2)$. The situation is even worse in the case of nonconfining mappings where no simplifications occur and the growth rate is maximal.

Since a considerable volume of results on integrable discrete systems were obtained with the method of singularity confinement it is natural to question their integrability in the light of the findings concerning the non sufficiency of the singularity confinement criterion. One of the aims of this review is to provide the confirmation of these results based on the more stringent criterion of algebraic entropy. In particular we shall examine several discrete Painlevé equations that we have obtained over the years and obtain their growth properties. The main result of this analysis is that, in every case studied, the singularity confinement criterion *when used for deautonomisation of an integrable mapping*, turns out to be sufficient [10]. This means that an equation, the autonomous form of which is integrable and which has been deautonomised following the constraints provided

by singularity confinement, leads to exactly the same degree-growth as the initial, autonomous, one. Thus we expect the deautonomised equation to be as integrable as its autonomous limit. This finding is of capital importance since it validates the results previously obtained with the singularity confinement approach. Moreover it sets the frame for a new, composite, approach. Whenever we wish to obtain the integrable cases of an equation containing a certain number of parameters we can perform a first, exploratory, study using the confinement method. Once the confinement constraints have been used in order to limit the freedom, the algebraic entropy, or low-growth criterion, can be implemented in order to pin down the truly integrable cases. Examples of this dual approach can be found in [11].

2. Discrete Painlevé equations

Let us first recall what has always been our approach to the derivation of discrete Painlevé equations. We start from an autonomous system the integrability of which has been independently established. In the case of discrete Painlevé equations, this system is the QRT mapping [9]:

$$f^{(1)}(x_n) - (x_{n+1} + x_{n-1})f^{(2)}(x_n) + x_{n+1}x_{n-1}f^{(3)}(x_n) = 0 \tag{2.1}$$

When the $f^{(i)}$'s are quartic functions, satisfying specific constraints, the mapping (2.1) is integrable in terms of elliptic functions. Since the elliptic functions are the autonomous limits of the Painlevé transcendents, the mapping (2.1) is the appropriate starting point for the construction of the nonautonomous discrete systems which are the analogues of the Painlevé equations. The procedure we used, often referred to as 'deautonomisation', consists in finding the dependence of the coefficients of the quartic polynomials appearing in (2.1) with respect to the independent variable n, which is compatible with the singularity confinement property. Namely, the n-dependence is obtained by asking that the singularities are indeed confined. The reason why this procedure can be justified is the following. Since the autonomous starting point is integrable, it is expected that the growth of the degree of the iterates is polynomial. Now it turns out that the application of the singularity confinement deautonomisation corresponds to the requirement that the nonautonomous mappings lead to the same factorizations and subsequent simplifications and have precisely the same growth properties as the autonomous ones. These considerations will be made more transparent thanks to the examples we present in what follows.

Let us start with a simple case. We consider the mapping:

$$x_{n+1} + x_{n-1} = \frac{a}{x_n} + \frac{1}{x_n^2} \tag{2.2}$$

where a depends on n. The singularity confinement result is that a must satisfy the conditions $a_{n+1} - 2a_n + a_{n-1} = 0$ i.e. a is linear in n. Assuming now that a is an arbitrary function of n we compute the iterates of (2.2). We obtain the sequence:

$$x_2 = \frac{q^2 + a_1pq - p^2r}{p^2}, \quad x_3 = \frac{pQ_4}{q(q^2 + a_1pq - p^2r)^2},$$
$$x_4 = \frac{(q^2 + a_1pq - p^2r)Q_7}{pQ_4^2}, \quad x_5 = \frac{pQ_4Q_{12}}{q(q^2 + a_1pq - p^2r)Q_7^2}$$

where the Q_k's are homogeneous polynomials in q, r of degree k. The simplifications that do occur are insufficient to curb the asymptotic growth. As a matter of fact, if we follow a particular factor we can check that it keeps appearing either in the numerator or the denominator (where its degree is alternatively 1 and 2). This corresponds to the unconfined singularity pattern $\{0, \infty^2, 0, \infty, 0, \infty^2, 0, \infty, \dots\}$. The confinement condition $a_{n+1} - 2a_n + a_{n-1} = 0$ is the condition for q to divide exactly Q_7, for both q and $r^2 + a_1 qr - pq^2$ to divide exactly Q_{12}, etc. Let us now turn to the computation of the degrees of x_n. We obtain successively 0, 1, 2, 5, 10, 21, 42, 85, ... The growth is exponential, the degrees behaving like $d_{2m-1} = (2^{2m} - 1)/3$ and $d_{2m} = 2d_{2m-1}$, a clear indication that the mapping is not integrable in general. Already at the fourth iteration the degrees differ in the autonomous and nonautonomous cases. Our approach consists in requiring that the degree in the nonautonomous case be *identical* to the one obtained in the autonomous one. If we implement the requirement that d_4 be 8 instead of 10 we find the condition $a_{n+1} - 2a_n + a_{n-1} = 0$, i.e. precisely the one obtained through singularity confinement. Moreover, once this condition is satisfied, the subsequent degrees of the nonautonomous case coincide with that of the autonomous one. Thus this mapping, leading to polynomial growth, should be integrable, and, in fact, it is. As we have shown in [9], where we presented its Lax pair, equation (2.2) with $a_n = \alpha n + \beta$ is a discrete form of the Painlevé I equation. In the examples that follow, we shall show that in all cases the nonautonomous form of an integrable mapping obtained through singularity confinement leads to exactly the same degrees of the iterates as the autonomous one.

We now turn to what is known as the "standard" discrete Painlevé equations [12] and compare the results of singularity confinement to those of the algebraic entropy approach. We start with d-P_I in the form:

$$x_{n+1} + x_n + x_{n-1} = a + \frac{b}{x_n}. \tag{2.3}$$

The degrees of the iterates of the autonomous mapping are 0, 1, 2, 3, 6, 9, 12, 17, 22, ... , i.e. a quadratic growth with $d_{3m+k}{=}3m^2 + (2m+1)k$, for k =0,1,2, while those of the generic nonautonomous one are 0, 1, 2, 3, 6, 11, ... Requiring two extra factorisations at that level (so as to bring d_5 down to 9) we find the following conditions $a_{n+1} = a_n$, so a_n must be a constant, and $b_{n+2} - b_{n+1} - b_n + b_{n-1}{=}0$, i.e. b_n is of the form $b_n = \alpha n + \beta + \gamma(-1)^n$ which are exactly the result of singularity confinement. Implementing these conditions we find that the autonomous and nonautonomous mappings have the same (polynomial) growth [10]. Both are integrable, the Lax pair of the nonautonomous one, namely d-P_I having been given in [13,14,15].

For the discrete P_{II} equation we have

$$x_{n+1} + x_{n-1} = \frac{a x_n + b}{x_n^2 - 1} \tag{2.4}$$

The degrees of the iterates in the autonomous case are d_n=0, 1, 2, 4, 6, 9, 12, 16, 20, ... , (i.e. $d_{2m-1}{=}m^2$, $d_{2m}{=}m^2 + m$) while in the generic nonautonomous case we find the first discrepancy for d_4 which is now 8. To bring it down to 6 we find two conditions, $a_{n+1} - 2a_n + a_{n-1}{=}0$ and $b_{n+1} = b_{n-1}$. This means that a is linear in n and b is an even/odd constant, as predicted by singularity confinement. Once we implement these constraints, the degrees of the nonautonomous and autonomous

cases coincide. The Lax pair of equation (2.4) in the nonautonomous form, i.e. d-P_{II}, has been presented in [13,16,17].

The q-P_{III} equation was obtained from the deautonomisation of the mapping:

$$x_{n+1}x_{n-1} = \frac{(x_n - a)(x_n - b)}{(1 - cx_n)(1 - x_n/c)} \tag{2.5}$$

In the autonomous case we obtain the degrees d_n=0, 1, 2, 5, 8, 13, 18, ... , just like for equation (2.2), while in the generic nonautonomous case we have 0, 1, 2, 5, 12, ... For d_4 to be 8 instead of 12, one needs four factors to cancel out. The conditions are $c_{n+1} = c_{n-1}$ and $a_{n+1}b_{n-1} = a_{n-1}b_{n+1} = a_n b_n$. Thus c is a constant up to an even/odd dependence, while a and b are proportional to λ^n for some λ, with an extra even/odd dependence, just as predicted by singularity confinement in [12]. The Lax pair for q-P_{III} has been presented in [13,18].

For the remaining three discrete Painlevé equations the Lax pairs are not known yet. It is thus important to have one more check of their integrability provided by the algebraic entropy approach. We start with d-P_{IV} in the form:

$$(x_{n+1} + x_n)(x_{n-1} + x_n) = \frac{(x_n^2 - a^2)(x_n^2 - b^2)}{(x_n + z_n)^2 - c^2} \tag{2.6}$$

where a, b and c are constants. If z_n is constant we obtain for the degrees of the successive iterates d_n=0, 1, 3, 6, 11, 17, 24, ... The general expression of the growth is d_n=$6m^2$ if $n = 3m$, d_n=$6m^2 + 4m + 1$ if $n = 3m + 1$ and d_n=$6m^2 + 8m + 3$ if $n = 3m + 2$. This polynomial (quadratic) growth is expected since in the autonomous case this equation is integrable, its solution being given in terms of elliptic functions. For a generic z_n we obtain the sequence d_n=0, 1, 3, 6, 13, ... The condition for the extra factorizations to occur in the last case, bringing down the degree d_4 to 11, is for z to be linear in n. We can check that the subsequent degrees coincide with those of the autonomous case.

For the discrete Painlevé V equation we start from:

$$(x_{n+1}x_n - 1)(x_{n-1}x_n - 1) = \frac{(x_n^2 + ax_n + 1)(x_n^2 + bx_n + 1)}{(1 - z_n cx_n)(1 - z_n dx_n)} \tag{2.7}$$

where a, b, c and d are constants. If moreover z is also a constant, we obtain exactly the same sequence of degrees d_n=0, 1, 3, 6, 11, 17, 24, ... , as in the d-P_{IV} case. Again, this polynomial (quadratic) growth is expected since this mapping is also integrable in terms of elliptic functions. For the generic nonautonomous case we again find the sequence d_n=0, 1, 3, 6, 13, ... Once more we require a factorization bringing down d_4 to 11. It turns out that this entails a z which is exponential in n, which then generates the same sequence of degrees as the autonomous case. In both the d-P_{IV} and q-P_{V} cases we find the n-dependence already obtained through singularity confinement. Since this results to a vanishing algebraic entropy we expect both equations to be integrable.

The final system we shall study is the one related to the discrete $\mathrm{P}_{\mathrm{V}I}$ equation:

$$\frac{(x_{n+1}x_n - z_{n+1}z_n)(x_{n-1}x_n - z_{n-1}z_n)}{(x_{n+1}x_n - 1)(x_{n-1}x_n - 1)} = \frac{(x_n^2 + az_nx_n + z_n^2)(x_n^2 + bz_nx_n + z_n^2)}{(x_n^2 + cx_n + 1)(x_n^2 + dx_n + 1)} \tag{2.8}$$

where a, b, c and d are constants. In fact the generic symmetric QRT mapping can be brought to the autonomous (z_n constant) form of equation (2.8) through

the appropriate homographic transformation. In the autonomous case, we obtain the degree sequence d_n=0, 1, 4, 9, 16, 25, ... , i.e. d_n=n^2. Since mapping (2.8) is rather complicated we cannot investigate its full freedom. Still we were able to perform two interesting calculations. First, assume that in the rhs instead of the function z_n a different function ζ_n appears. In this case the degrees grow like 0, 1, 5, ... , and the condition to have d_2=4 instead of 5 is $z_{n+1}z_{n-1}z_n^2 = \zeta_n^4$. Assuming this is true, we compute the degree d_3 of the next iterate and find d_3=13 instead of 9. To bring down d_3 to the value 9 we need $z_n^2 = \zeta_n^2$, which up to a redefinition of a and b means $z_n = \zeta_n$. This implies $z_{n+1}z_{n-1} = z_n^2$, and z_n is thus an exponential function of n, z_n=$z_0\lambda^n$ (which is in agreement with the results of [19]). Then a quartic factor drops out and d_3 is just 9. One can then check that the next degree is 16, just as in the autonomous case. Thus the q-$\mathrm{P}_{\mathrm{V}I}$ equation leads to the same growth as the generic symmetric QRT mapping and is thus expected to be integrable. As a matter of fact we were able to show that the generic *asymmetric* QRT mapping leads to the same growth d_n=n^2 as the symmetric one. This is not surprising, given the integrability of this mapping. What is interesting is that the growth of the generic symmetric and asymmetric QRT mappings are the same. Thus d_n=n^2 is the maximal growth one can obtain for the QRT mapping in the homogeneous variables we are using. As a matter of fact we have also checked that the asymmetric nonautonomous q-$\mathrm{P}_{\mathrm{V}I}$ equation, introduced in [19] led to exactly the same degree-growth d_n=n^2.

The singularity confinement results have been confronted to the algebraic entropy approach for several other discrete Painlevé equations. In every case examined the deautonomisation obtained has turned out to be the right one, it was the condition for the degree-growth to be identical to the one of the autonomous system. Thus despite the non sufficiency of the singularity confinement the integrability predictions for discrete Painlevé equations based on this criterion are confirmed.

3. Linearisable equations

In this section we shall examine this particular class of mappings which are linearisable and study their growth properties. Most of these systems were obtained using the singularity confinement criterion and thus a study of the growth of the degree of the iterates would be an interesting complementary information. Moreover, as we will show, the linearisable systems do possess particular growth properties which set them apart from the other integrable discrete systems.

The first mapping we are going to treat is a two-point mapping of the form $x_{n+1} = f(x_n, n)$ where f is rational in x_n and analytical in n. As explained in the introduction for all f's of the form $\sum_k \frac{\alpha_k}{(x_n+\beta_k)^{\nu_k}}$ the singularity confinement requirement is satisfied. However only the discrete Riccati, $x_{n+1} = \alpha + \frac{\lambda}{x_n+\beta}$, is expected to be integrable. Our argument in [20], for the rejection of these confining but nonintegrable cases, was based on the proliferation of the preimages of a given point. If we solve the mapping for x_n in terms of x_{n+1} we do not find a uniquely defined x_n and, iterating, the number of x_{n-k} grows exponentially. In what follows we shall analyse this two-point mapping in the light of the algebraic entropy approach. We start from the simplest case which we expect to be nonintegrable,

$$x_{n+1} = \alpha + \frac{\lambda}{x_n + \beta} + \frac{\mu}{x_n + \gamma}. \qquad (3.1)$$

The initial condition we are going to iterate is $x_0 = p/q$ and the degree we calculate is the homogeneous degree in p and q of the numerator (or the denominator) of the iterate. We obtain readily the following degree sequence $d_n = 1, 2, 4, 8, 16, \ldots$ i.e. $d_n = 2^n$. Thus the algebraic entropy of the mapping is $\log(2) > 0$, an indication that the mapping cannot be integrable. Now we ask how can one curb the growth and make it nonexponential. It turns out that the only possibilities are $\lambda\mu = 0$ or $\beta = \gamma$. In either case mapping (3.1) becomes a homography. The degree in this case is simply $d_n = 1$ for all n. This is an interesting result, clearly due to the fact that the homographic mapping is linearisable through a simple Cole-Hopf transformation.

The second mapping we shall examine is one due to Bellon and collaborators [21]

$$\begin{aligned} x_{n+1} &= \frac{x_n + y_n - 2x_n y_n^2}{y_n(x_n - y_n)}, \\ y_{n+1} &= \frac{x_n + y_n - 2x_n^2 y_n}{x_n(y_n - x_n)}. \end{aligned} \tag{3.2}$$

The degree-growth in this case is studied starting from $x_0 = r$, $y_0 = p/q$ and again we calculate the homogeneous degree of the iterate in p and q, i.e. we set the degree of r to zero. (Other choices could have been possible but the conclusion would not depend on these details.) We obtain the degrees $d_{x_n} = 0, 2, 2, 4, 4, 6, 6, \ldots$ and $d_{y_n} = 1, 1, 3, 3, 5, 5, \ldots$ i.e. a linear degree-growth. This is in perfect agreement with the integrable character of the mapping. As was shown in [22] it does satisfy the unique preimage requirement and possesses a constant of motion $k = \frac{1 - x_n y_n}{y_n - x_n}$, the use of which reduces it to a homographic mapping for x_n or y_n.

The third mapping we are going to study is the one proposed in [20]

$$\begin{aligned} x_{n+1} &= \frac{x_n(x_n - y_n - a)}{x_n^2 - y_n}, \\ y_{n+1} &= \frac{(x_n - y_n)(x_n - y_n - a)}{x_n^2 - y_n} \end{aligned} \tag{3.3}$$

where a was taken constant. We start by assuming that a is an arbitrary function of n and compute the growth of the degree. We find $d_{x_n} = 0, 1, 2, 3, 4, 5, 6, 7, 8, \ldots$ and $d_{y_n} = 1, 2, 3, 4, 5, 6, 7, 8, 9, \ldots$ i.e. again a linear growth. This is an indication that (3.3) is integrable for arbitrary a_n and indeed it is. Dividing the two equations we obtain $y_{n+1}/x_{n+1} = 1 - y_n/x_n$ i.e. $y_n/x_n = 1/2 + k(-1)^n$ whereupon (3.3) is reduced to a homographic mapping for x. Thus in this case the degree-growth has successfully predicted integrability.

A picture starts emerging at this point. While in our study of discrete Painlevé equations and the QRT mapping we found quadratic growth of the degree of the iterate, linearisable second-order mappings seem to lead to slower growth. In order to investigate this property in detail we shall analyse the three-point mapping we have studied in [4,23] from the point of view of integrability in general and linearisability in particular. The generic mapping studied in [23] was one trilinear in x_n, x_{n+1}, x_{n-1}. Several cases were considered. Our starting point is the mapping,

$$x_{n+1}x_nx_{n-1} + \beta x_nx_{n+1} + \zeta\eta x_{n+1}x_{n-1} + \gamma x_nx_{n-1} + \beta\gamma x_n + \eta x_{n-1} + \zeta x_{n+1} + 1 = 0. \tag{3.4}$$

We start with the initial conditions $x_0 = r$, $x_1 = p/q$ and compute the homogeneous degree in p, q at every n. We find $d_n = 0, 1, 1, 2, 3, 5, 8, 13, \dots$ i.e. a Fibonacci sequence $d_{n+1} = d_n + d_{n-1}$ leading to exponential growth of d_n with asymptotic ratio $\frac{1+\sqrt{5}}{2}$. Thus mapping (3.4) is not expected to be integrable in general. However, as shown in [23] integrable subcases do exist. We start by requiring that the degree-growth be less rapid and as a drastic decrease in the degree we demand that $d_3 = 1$ instead of 2. We find that this is possible when either $\beta = \zeta = 0$ in which case the mapping reduces to:

$$x_{n+1} = -\gamma - \frac{\eta}{x_n} - \frac{1}{x_n x_{n-1}} \tag{3.5}$$

or $\gamma = \eta = 0$, giving a mapping identical to (3.5) after $x \to 1/x$. In this case the degree is $d_n = 1$ for $n > 0$. Equation (3.5) is the generic projective three-point mapping, written in canonical form. Its linearisation can be obtained [23] in terms of a system of three linear equations, a fact which explains the constancy of the degree.

Non generic subcases of (3.4) some of which are integrable do exist. They have been studied in [23,24].

Linearisability through the reduction to a projective system is not the only possibility. Other possibilities do exist. Let us start by considering the generic three-point mapping that can be considered as the discrete derivative of a (discrete) Riccati equation. Let us start from the general homographic mapping which we can write as

$$Ax_n x_{n+1} + Bx_n + Cx_{n+1} + D = 0 \tag{3.6}$$

where A, B, C, D are linear in some constant quantity κ. In order to take the discrete derivative we extract the constant κ and rewrite (3.6) as:

$$\kappa = \frac{\alpha x_n x_{n+1} + \beta x_n + \gamma x_{n+1} + \delta}{\epsilon x_n x_{n+1} + \zeta x_n + \eta x_n up + \theta}. \tag{3.7}$$

Using the fact that κ is a constant, it is now easy to obtain the discrete derivative by downshifting (3.7) and subtracting it form (3.7) above. Instead of examining this most general case we concentrate on the forms proposed in [25]. They correspond to the reduction of (3.7) to the two cases:

$$\kappa = x_{n+1} + a + \frac{b}{x_n} \tag{3.8}$$

$$\kappa = \frac{x_{n+1}(x_n + a)}{x_n + b} \tag{3.9}$$

Next we compute the discrete derivatives of (3.8) and (3.9). We find:

$$x_{n+1} = x_n + a_{n-1} - a_n - \frac{b_n}{x_n} + \frac{b_{n-1}}{x_{n-1}} \tag{3.10}$$

and

$$x_{n+1} = x_n \frac{x_{n-1} + a_{n-1}}{x_n + a_n} \frac{x_n + b_n}{x_{n-1} + b_{n-1}}. \tag{3.11}$$

The study of the degree of growth of (3.10) and (3.11) can be performed in a straightforward way. For both mappings we find the sequence $d_n = 0, 1, 2, 3, 4, 5, 6, \dots$ i.e. a linear growth just as in the cases of mappings (3.2) and (3.3). If we substitute b_{n-1} by c_{n-1} in the last term of the rhs of (3.10) or the denominator of (3.11) we

find $d_n = 0, 1, 2, 4, 8, 16, \ldots$ i.e. $d_n = 2^n$ for $n > 0$ unless $c = b$. Investigating all the possible ways to curb the growth we find for both (3.10) and (3.11) that $c = 0$ is also a possibility to bring d_3 down to 3. However a detailed analysis of this case shows that for $c = 0$ we have $d_n = 0, 1, 2, 3, 5, 8, 13, 21, \ldots$ i.e. a Fibonacci sequence with slower, but still exponential, growth (i.e. ratio $\frac{1+\sqrt{5}}{2}$ instead of 2).

A bonus study: the N-dimensional Gambier mapping

While most of the results included in this paper were presented in previous publications of ours, in this section we shall present a study which appear for the first time in these proceedings. It concerns the extension of the Gambier mapping to N dimensions.

The two-dimensional Gambier mapping was introduced in [26,27] as a discretisation of the second order differential equation discovered by Gambier, in his study of second order ODE's having the Painlevé property. The Gambier equation is in fact a system of two Riccati's in cascade. The latter means that the system consists in one Riccati for one variable and a second one (for the second variable) with coefficients depending (linearly) on the variable of the first one. The discretisation of the Gambier system is straightforward: just substitute homographic mapping, i.e. discrete Riccati, in the previous sentence. Thus the general second-order Gambier mapping is given by:

$$y_{n+1} = \frac{ay_n + b}{cy_n + d}, \tag{3.12a}$$

$$x_{n+1} = \frac{(ey_n + f)x_n + (gy_n + h)}{(jy_n + k)x_n + (ly_n + m)} \tag{3.12b}$$

where a, b, $\ldots$, m are functions of n. In [27] we have studied this mapping in detail from the point of view of the singularity structure. This was motivated by the fact that we aimed at being able to express the solution as an infinite product of matrices, *even across singularities*. On the other hand if we are not interested in this fine point, the linearisability of (3.12) can be obtained through a Cole-Hopf transformation for each variable.

The study of the degree-growth of (3.12) is straightforward. We start from $x_0 = r$, $y_0 = p/q$ and compute the homogeneous in p, q degree of (4.1a) and (4.1b). Since (3.12a) is a Riccati its degree does not grow i.e. we have $d_{y_n} = 1$. Given the structure of (3.12b) we have $d_{x_{n+1}} = d_{x_n} + d_{y_n}$ and thus $d_{x_n} = n$. What is interesting here is that the Gambier mapping exhibits a linear degree-growth independently of the precise values of a, b, $\ldots$, m. The fact that it can be reduced to Riccati's in cascade is enough to guarantee its integrability.

The generalisation of (3.12) to $N+1$ dimensions is straightforward. We find the system:

$$\begin{aligned} x_{n+1}^{(0)} &= \frac{a^{(0)}x_n^{(0)} + b^{(0)}}{c^{(0)}x_n^{(0)} + d^{(0)}}, \\ x_{n+1}^{(i)} &= \frac{(e^{(i)}x_n^{(i-1)} + f^{(i)})x_n^{(i)} + (g^{(i)}x_n^{(i-1)} + h^{(i)})}{(j^{(i)}x_n^{(i-1)} + k^{(i)})x_n^{(i)} + (l^{(i)}x_n^{(i-1)} + m^{(i)})}, \quad i = 1, \ldots, N. \end{aligned} \tag{3.13}$$

The study of the degree-growth of system (3.13) can be performed along the $N = 1$ case. From (3.13) we have the recurrence: $dx_{n+1}^{(k)} = dx_n^{(k)} + dx_n^{(k-1)}$. We obtain formally $dx_{n+1}^{(k)} = \sum_{p=0}^{n} dx_p^{(k-1)}$. Thus starting from $dx_n^{(0)} = 1$ we find

$dx_n^{(N)} \propto n^N$. Thus the $(N+1)$-dimensional Gambier mapping has polynomial growth for every dimension. The linearisation is obtained through a Cole-Hopf transformation.

Given the structure of (3.13) it is clear that we can solve successively for each variable and express finally (3.13) as a single $(N+2)$-point mapping. This leads us to another integrable discretisation of linearisable mappings with cascade structure. Let us consider the second order system

$$\begin{aligned} y_{n+1} &= \frac{ay_n + b}{cy_n + d}, \\ x_{n+1} &= \frac{f_1(y_n)x_n - f_2(y_n)}{f_4(y_n) - f_3(y_n)x_n} \end{aligned} \tag{3.14}$$

where the f_i's are polynomial in y_n. From the structure of (3.14) it is clear that it can be linearised, independently of the precise form of the f_i. Similarly the degree-growth of x_n is always linear. On the other hand it is not, in general, possible to express (3.14) as a single three-point mapping. Extension of (3.14) to N-dimensions can be obtained along the lines of the N-dimensional discrete Gambier system.

In this section we have studied the growth properties of various linearisable systems identified through the singularity confinement criterion. In every case the singularity confinement results were confirmed. Moreover it turned out that the linearisable systems lead to slower growth than systems which are integrable by other methods. This property could be used for the classification of integrable systems and be a valuable indication as to the precise method of their integration.

4. Conclusion

In this work we have presented a comparative review of results obtained with the singularity confinement and the algebraic entropy methods. While the former approach is not a sufficient criterion of integrability, the confirmation of its results (through the second, more stringent, criterion) in the domain of discrete Painlevé equations leads to interesting new insights. In every case examined, the singularity confinement, when used for the deautonomisation of an integrable autonomous mapping turned out to give sufficient constraints for the degree-growth to be non-exponential. Thus we can propose a new strategy for the detection of integrable discrete systems. Given a mapping which contains several parameters the singularity confinement necessary criterion can be used in order to screen it for possible integrable discrete cases. Once the research domain is reduced the growth properties can be studied leading to better integrability candidates.

Another interesting result of our studies concerns the degree-growth of linearisable systems. We found that, while for second-order mappings the generic integrable case is associated to quadratic growth, the linearisable mappings lead to zero or linear growth. The growth exponent is of course a property which depends on the dimension. We surmise that the generic integrable Nth order mapping will lead to growth n^N, while the linearisable mappings of the same dimension will lead to slower growth. Our study of the Nth order Gambier system shows that the growth is n^{N-1}, and we expect the the projective system of order N to lead to zero growth. Thus the detailed study of the growth properties can become a precious indication as to the precise method of integration of a given discrete system.

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